PMM U.S.S.R.,Vol.51,No.3,pp.346-353,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

FINITE STRAINS OF VISCOELASTIC MUSCLE TISSUE*

V.I. KONDAUROV and L.V. NIKITIN

The mechanical behaviour of muscle tissue whose characteristic feature is the capacity to contract under the effect of factors of non-mechanical nature, is investigated.

This research is an extension and development of the phenomenological model of the biologically active behaviour of a material under the joint action of mechanical forces and a signal of electrochemical nature /1/ in the finite-strain case. In addition, relationships of the relaxation type are used for both the viscous and biological strain components. Consequently, the current state of the muscle is determined by the prehistory of the strain and stimulation. The investigation is made in an isothermal approximation for a one-component transversally isotropic homogeneous medium. The constraints that impose invariance requirements with respect to certain body configurations on the structure of the rheological relationships are examined. Concepts of an ideal and reinforcing biologically active viscoelastic material are introduced on their basis.

The model of a muscle as a multiphase medium with chemical interaction between the phases was developed in /2-7/**. (**See also: Tsaturyan, A.K., A continuum model of the heart muscle and its experimental confirmation. Report No.2525, Scientific Research Institute of Mechanics, Moscow State University, 1981). Consequently, governing equations are obtained for muscle that are similar to /1/, and a decoding is given for the biological strain in terms of the model parameters. The dependence of biological strain on prehistory was taken into account in a linear approximation in /8/.

The important case, from the viewpoint of applications, of small elastic and finite viscous and biological strains is considered in detail. It is shown that the approach utilized yields a single-valued solution of the problem of selecting the objective derivative in the rheological relationships that connect the rate of change of the stress tensor with the strain rates. As an illustration we consider the axisymmetric problem of strain under the action of internal pressure on a cylinder from an ideal biologically active viscoelastic material.

1. Kinematics.

We utilize a representation of the total strain in the form of a composition of biological, viscous, and elastic strains to describe the large strains of a continuous medium that possesses viscoelastic properties and responds to a signal of electrochemical nature. Let $\varkappa$ be the reference, χ the current body configuration and $\mathbf{X} \in \varkappa$, $\mathbf{x} \in \chi$ the radius-vectors of body particles in these configurations. As usual, we consider the mapping $\varkappa \rightarrow \chi$ to be non-degenerate and sufficiently smooth. Let $\mathbf{F}$ be the gradient of this mapping, i.e.,

$$d\mathbf{x} = \mathbf{F} d\mathbf{X}, \quad \det \mathbf{F} \neq 0 \tag{1.1}$$

In addition to $\varkappa$ and χ we introduce the intermediate configurations $\varkappa_1$ and $\varkappa_2$ belonging to non-Euclidean space in the general case. If $\mathbf{X}_1$ and $\mathbf{X}_2$ are radius-vectors of particles in the configurations $\varkappa_1(\mathbf{X})$, $\varkappa_2(\mathbf{X})$ tangent to $\varkappa_1$ and $\varkappa_2$ at the point $\mathbf{X}$ while $\mathbf{B}, \mathbf{P}$ and $\mathbf{E}$ are the gradients of non-degenerate mappings $\varkappa \rightarrow \varkappa_1(\mathbf{X}), \varkappa_1(\mathbf{X}) \rightarrow \varkappa_2(\mathbf{X}), \varkappa_2(\mathbf{X}) \rightarrow \chi$, respectively, then

$$d\mathbf{x} = \mathbf{E}\, d\mathbf{X}_2, \quad d\mathbf{X}_2 = \mathbf{P}\, d\mathbf{X}_1, \quad d\mathbf{X}_1 = \mathbf{B}\, d\mathbf{X}, \quad \det \mathbf{E} \neq 0, \quad \det \mathbf{P} \neq 0, \quad \det \mathbf{B} \neq 0 \tag{1.2}$$

It follows from (1.1) and (1.2)

$$\mathbf{F} = \mathbf{EPB} \tag{1.3}$$

i.e., the total strain gradient $\varkappa \rightarrow \chi$ is the product of the mapping gradients $\varkappa_2 \rightarrow \chi$, $\varkappa_1 \rightarrow \varkappa_2$, and $\varkappa \rightarrow \varkappa_1$, which we will designate the elastic, viscous, and biological strains.

*Prikl.Matem.Mekhan.,51,3,443-452,1987

Relationships (1.1)-(1.3) are a natural extension of the description of the kinematics of a biologically active body proposed in /1/ for small strains of an elastic body, taking the so-called biofactor into account, to the finite strain case.

It follows from the non-degeneracy of all the mappings that definite polar expansions hold in a natural manner

$$\mathbf{F}=\mathbf{R}_F\mathbf{U}_F,\quad \mathbf{E}=\mathbf{R}_E\mathbf{U}_E,\quad \mathbf{P}=\mathbf{R}_P\mathbf{U}_P,\quad \mathbf{B}=\mathbf{R}_B\mathbf{U}_B \tag{1.4}$$

where $\mathbf{R}_F, \mathbf{R}_E, \mathbf{R}_P, \mathbf{R}_B$ are orthogonal, and $\mathbf{U}_F, \mathbf{U}_E, \mathbf{U}_P, \mathbf{U}_B$ are symmetric positive-definite tensors.

We denote the particle velocity vector by $\mathbf{v} = \partial \mathbf{x}/\partial t\ |_X$ and we recall the well-known kinematic relationship

$$\mathbf{F}^{\cdot}\mathbf{F}^{-1} = \nabla\mathbf{v} \tag{1.5}$$

Here and henceforth, the dot denotes the derivative with respect to t for $X = \text{const}$ and ∇ is the gradient in the variables $\mathbf{x}$.

We stress the following features of the kinematics of the medium being studied. Firstly, the representation $\mathbf{F}$ in the form of the composition (1.3) does not mean that the total strain is ralized by successive development in time of first the biological, and then the viscous and the elastic. All three components develop simultaneously. The order of tracking the gradients $\mathbf{E}, \mathbf{P}, \mathbf{B}$ in (1.3) is not important, the model can be formulated with equal success on the basis of another decomposition. Secondly, the orthogonal tensors $\mathbf{R}_P, \mathbf{R}_B$ characterizing the rotation of the intermediate configurations enter into the composition (1.3) in addition to the "pure" strains $\mathbf{U}_E, \mathbf{U}_P, \mathbf{U}_B$. Investigation of the general structure of the rheological relationships shows that the properties of the medium are determined by combinations of quantities that depend on both $\mathbf{U}_E, \mathbf{U}_P, \mathbf{U}_B$ and on $\mathbf{R}_P, \mathbf{R}_B$.

It should also be noted that the representation (1.3) does not contain any partition "rules" and even constraints on the values of the non-degenerate tensors of second rank $\mathbf{E}$, $\mathbf{P}$, $\mathbf{B}$. The situation here is completely analogous to the kinematics of elastic-plastic bodies in the sense that it is impossible to speak about the magnitudes of the strain components until the governing relationships, the conservation laws, and the boundary conditions have been taken into account.

2. Rheological relationships.

We limit ourselves to an isothermal approximation. We will consider the elastic potential A and the Cauchy stress tensor $\mathbf{T}$ to have the form

$$A = A^+(\mathbf{F}, \mathbf{P}, \mathbf{B}, f),\quad \mathbf{T} = \rho(\partial A^+/\partial\mathbf{F})\,\mathbf{F}^T = \mathbf{T}^+(\mathbf{F}, \mathbf{P}, \mathbf{B}, f) \tag{2.1}$$

where ρ is the material density in the current configuration. Among the arguments of the functions A^+ and $\mathbf{T}^+$ in addition to the strain characteristics is the scalar quantity $f(\mathbf{X}, t)$ that gives the stimulating signal that realizes the central regulation of the muscle activity.

We will caracertize the viscous properties of the material by the relation

$$\mathbf{\Phi}(\mathbf{P}^{\cdot}, \mathbf{F}, \mathbf{P}, \mathbf{B}, f) = 0 \tag{2.2}$$

which yields the rate of production of the viscous strains. Therefore, the viscous strain is described by a functional since it depends on the prehistory of the arguments in (2.2). The particular form taken for the functional (2.2) describes a viscoelastic material of so-called relaxation type.

The biological strain $\mathbf{B}$ is produced by the stimulating signal $f(\mathbf{X}, t)$. In order to outline this principal property of active biological strain, it was defined in /1, 9/ as the strain caused by a signal in the absence of mechanical action, and therefore, is independent of the running state of the muscle. The same idea was propounded in /10/. However, the feedback between the signal and the running state of the muscle, i.e., the local regulation of the muscle tissue, exists /11/, in addition, the elastic and relaxation properties of the material can depend on f /12/. Therefore, for generality, the dependence of the biological strain on $\mathbf{F}$ and $\mathbf{P}$ should be taken into account. However, the fact that the state of muscle stress and strain depends very much on the prehistory of the stimulating signal, as follows from numerous experiments /11-13/, is most important.

We will take a governing relationship of relaxation type

$$\mathbf{\Gamma}(\mathbf{B}^{\cdot}, \mathbf{F}, \mathbf{P}, \mathbf{B}, f) = 0 \tag{2.3}$$

for the biological strain.

The lack of an explicit dependence on $\mathbf{X}, t$ in (2.1)-(2.3) indicates that for $f \equiv 0$ the governing equations will be invariant under shear in the time t and the particle $\mathbf{X}$, i.e., the material in the passive state is a homogeneous, non-ageing viscoelastic material whose rheology is included in the general theory of governing equations /14, 15/. Since the signal f depends on $\mathbf{X}, t$ there is no invariance with respect to shear in $\mathbf{X}, t$ in the active state. This means that the material acquires inhomogeneity dependent on the time and the property of "ageing". Unlike media being considered in /16/, say, the "age" changes of such a material are non-monotonic and reversible in time.

The model of a biologically active continuous medium being considered can also be treated as the model of a traditional viscoelastic material formulated in the terminology of the "reference" configuration (RC), obtained from the natural biological strain configuration.

In the general case of inhomogeneous biological strains, such an RC belongs to a non-Euclidean space and is variable in time. This specifies a definite similarity of the model under consideration with the models of elastic-plastic and viscoelastic media with instantaneously elastic reactions, which are the theory of elastic bodies with RC in agreement with an unloaded configuration.

In addition to this similarity, a clear distinction exists that is apparently due to the energy aspect of the problem. If plasticity and viscosity are related only to the transformation of work and internal energy into heat, then the strains of a biologically active medium are accompanied by transformations of chemical energy. Hence, inelastic strains or a change in the RC in pasticity models are determined just by thermomechanical processes and in this sense the model is closed. Models of biologically active media of a purely mechanical or thermomechanical nature without considering the processes of chemical energy transformation into other kinds are not closed in principle and contain a certain aribtrariness in the form of the given external effect of a non-thermomechanical nature.

Let us now clarify the constraints that are imposed on the structure of the rheological relationships (2.1)-(2.3) by the requirements that the equations be invariant under certain transformations of the body configurations introduced. We will consider the following principles as valid for the media under consideration.

1°. The governing Eqs.(2.1)-(2.3) are invariant relative to the full orthogonal group of transformations of the actual configuration.

2°. The form of (2.1)-(2.3) is invariant under unimodular, $\mathbf{X}, t$ -dependent configurations tangent to intermediate configurations and including the full orthogonal transformation group.

3°. The governing relationships are invariant under the group of equivalent transformations of configurations tangent ot the RC.

A tangent configuration is understood to be a configuration of a homogeneously strained body in whose particles the strains and electrochemical signal are constant in space and equal to the strains and signal in the particle $\mathbf{X}$ under consideration. The introduction of such configurations is due to the local nature of the dependence of the governing equations on the RC mapping $\varkappa$ into the actual and intermediate.

Proposition 1° expresses a weakened principle of objectivity or independence of the governing relationships from the body motion as a rigid whole without considering a time shift /15, 17/. The physical meaning of requirement 2° is that for configurations tangent to $\varkappa_1$ and $\varkappa_2$ that belong to a Euclidean space and are the acutal body configurations subjected just to homogeneous biological or biological and viscous strain, independence of the governing equations is postulated as the minimum with respect to rotation of the body as a rigid whole. Condition 3° is the definition of the symmetry of an inhomogeneous material whose inhomogeneity is due to the signal $f(\mathbf{X}, t)$.

Henceforth we will confine ourselves to the case of transversally isotropic media for which an undistorted RC $\varkappa_0$ exists with an equivalency group $g_{\varkappa_0}$ whose elements are either orthogonal tensors $\mathbf{Q}_n$ describing the rotation around on axis given by the unit vector $\mathbf{n}$, or all unimodular tensors $\mathbf{K}_n$ of transversally isotropic strain that have the form

$$\mathbf{K}_n = \mathbf{Q}_n\mathbf{U}_n = \mathbf{U}_n\mathbf{Q}_n, \quad \mathbf{U}_n = (a - a^{-1/2})\,\mathbf{n}\otimes\mathbf{n} + a^{-1/2}\mathbf{I}, \quad a > 0 \tag{2.4}$$

in addition to $\{\mathbf{I}, -\mathbf{I}\}$, where $\mathbf{I}$ is a unit tensor of second rank.

This enables us to determine the two most interesting classes of transversally isotropic biologically active viscoelastic media. One is given by the relationship

$$g_{\varkappa_1} = g_{\varkappa_2} = o, \quad g_{\varkappa_0} = \{\mathbf{Q}_n\} \tag{2.5}$$

and will be called the reinforcing material.

The necessary and sufficient conditions for invariance of the governing equations (2.1)-(2.3) with respect to the transformation group (2.5) of the configurations $\varkappa_0, \varkappa_\alpha\ (\alpha = 1, 2)$ and invariance with respect to orthogonal transformations of the configuration χ are the conditions

$$A = A^\circ(\Lambda_0), \quad \mathbf{T} = \mathbf{R}_F\mathbf{T}^\circ(\Lambda_0)\,\mathbf{R}_F^T, \quad \mathbf{W}^\cdot = \boldsymbol{\Psi}^\circ(\Lambda_0), \quad \mathbf{U}_B^\cdot = \boldsymbol{\Omega}^\circ(\Lambda_0) \tag{2.6}$$

where $A^\circ, \mathbf{T}^\circ, \boldsymbol{\Psi}^\circ$ and $\boldsymbol{\Omega}^\circ$ are functions that are isotropic with respect to the full orthogonal group o, i.e.,

$$\begin{gathered}
A^\circ(\Lambda_0^Q) = A^\circ(\Lambda_0), \quad \mathbf{T}^\circ(\Lambda_0^Q) = \mathbf{Q}\mathbf{T}^\circ(\Lambda_0)\,\mathbf{Q}^T, \\
\boldsymbol{\Psi}^\circ(\Lambda_0^Q) = \mathbf{Q}\boldsymbol{\Psi}^\circ(\Lambda_0)\,\mathbf{Q}^T, \quad \boldsymbol{\Omega}^\circ(\Lambda_0^Q) = \mathbf{Q}\boldsymbol{\Omega}^\circ(\Lambda_0)\,\mathbf{Q}^T \\
\Lambda_0 = \{\mathbf{U}_F, \mathbf{W}, \mathbf{U}_B, f, \mathbf{n}\}, \quad \mathbf{W} = \mathbf{R}_B^T\mathbf{U}_P\mathbf{R}_B \\
\Lambda_0^Q = \{\mathbf{Q}\mathbf{U}_F\mathbf{Q}^T, \mathbf{Q}\mathbf{W}\mathbf{Q}^T, \mathbf{Q}\mathbf{U}_B\mathbf{Q}^T, f, \mathbf{Q}\mathbf{n}\}
\end{gathered} \tag{2.7}$$

($Q \Subset o$ is an arbitrary constant orthogonal tensor).

Under the assumption that f is an objective scalar, the proof of the necessity and sufficiency of conditions (2.6) and (2.7) is analogous to that presented in /18/ for the case of isotropic viscoelastic media $N \geqslant 1$ by intermediate configurations.

We will call a second, narrower class of media ideal transversally isotropic biologically active viscoelastic materials. It is assumed for such media that the biological strain is isochoric, i.e., $\mathbf{B}$ is a unimodular tensor while the elastic potential, stress, and rate of vioscous strain are explicitly independent of the signal f.

The transformation group for the configuration $\varkappa_0$ and $\varkappa_\alpha$ for which (2.1)-(2.3) are invariant, is postulated in the form

$$g_{\varkappa_2} = g_{\varkappa_1} = o, \quad g_{\varkappa_0} = \{\mathbf{K}_n\} \tag{2.8}$$

And we shall consider that the biological strain $\mathbf{U}_B$ does not alter the transversal isotropy of the material. This means that just as the tensor $\mathbf{U}_n$ given by (2.4), the tensor $\mathbf{U}_B$ is determined in terms of one scalar and has the form

$$\mathbf{U}_B = (\omega^2 - \omega^{-1})\,\mathbf{n} \otimes \mathbf{n} + \omega^{-1}\mathbf{I}, \quad \omega = \omega\,(\mathbf{X}, t) > 0 \tag{2.9}$$

The conditions

$$\begin{aligned} &A = A^+(\Lambda_+), \quad \mathbf{T} = \mathbf{R}\mathbf{T}^+(\Lambda_+)\,\mathbf{R}^T \\ &\mathbf{W}^\cdot = \Psi^+(\Lambda_+), \quad \omega^\cdot\omega^{-1} = \Omega^+(\Lambda_+, f) \end{aligned} \tag{2.10}$$

where A^+, $\mathbf{T}^+$, Ψ^+ and Ω^+ are functions isotropic with respect to the full orthogonal group o, i.e.,

$$\begin{aligned} &A^+(\Lambda_+{}^Q) = A^+(\Lambda_+), \quad \mathbf{T}^+(\Lambda_+{}^Q) = \mathbf{Q}\mathbf{T}^+(\Lambda_+)\,\mathbf{Q}^T, \\ &\Psi^+(\Lambda_+{}^Q) = \mathbf{Q}\Psi^+(\Lambda_+)\,\mathbf{Q}^T, \quad \Omega^+(\Lambda_+{}^Q, f) = \Omega^+(\Lambda_+, f) \\ &\Lambda_+ = \{\mathbf{V}, \mathbf{W}, \mathbf{n}\}, \quad \Lambda_+{}^Q = \{\mathbf{QVQ}^T, \mathbf{QWQ}^T, \mathbf{Qn}\}, \\ &\mathbf{V} = \mathbf{R}_B{}^T\mathbf{R}_P{}^T\mathbf{U}_E\mathbf{R}_P\mathbf{R}_B, \quad \mathbf{W} = \mathbf{R}_B{}^T\mathbf{U}_P\mathbf{R}_B, \quad \mathbf{R} = \mathbf{R}_E\mathbf{R}_P\mathbf{R}_B \end{aligned} \tag{2.11}$$

are necessary and sufficient conditions for indifference of the governing equations (2.1)-(2.3) with respect to orthogonal transformations of the actual configuration and invariance under the transformation group (2.8).

The proof of assertions (2.8) and (2.9) is analogous to that in /18/ as in the case of reinforcing materials.

3. Small elastic strains.

For brevity we use the notation $\mathbf{S} = \mathbf{W}\mathbf{U}_B$, $\mathbf{S} \neq \mathbf{S}^T$. The gradient of the total strain $\mathbf{F}$ can then be written in the form of the product $\mathbf{F} = \mathbf{RVS}$. We introduce the elastic strain tensor

$$\mathbf{C} = {}^1/_2\,(\mathbf{V}^2 - \mathbf{I}) = {}^1/_2\,(\mathbf{S}^{-1T}\mathbf{F}^T\mathbf{F}\mathbf{S}^{-1} - \mathbf{I}) \tag{3.1}$$

defined uniquely by the tensor $\mathbf{V}$ and agreeing with the Cauchy-Green strain tensor for $\mathbf{S} = \mathbf{I}$. It is possible to transfer from the arguments $(\mathbf{V}, \mathbf{W}, \mathbf{n})$ to the variables $(\mathbf{C}, \mathbf{W}, \mathbf{n})$ in the rheological relationships for an ideal material.

Taking account of (3.1) the relation $\mathbf{T} = \rho\,(\partial A/\partial \mathbf{F})\mathbf{F}^T$ between the stress tensor and the elastic potential can be reduced to the form

$$\begin{aligned} \rho^{-1}\mathbf{T} &= \mathbf{F}\mathbf{S}^{-1}\,(\partial A^+/\partial \mathbf{C})\,\mathbf{S}^{-1T}\mathbf{F}^T = \mathbf{RV}\,(\partial A^+/\partial \mathbf{C})\,\mathbf{V}\mathbf{R}^T = \\ &\mathbf{R}\,(\mathbf{I} + 2\mathbf{C})^{1/2}\,(\partial A^+/\partial \mathbf{C})\,(\mathbf{I} + 2\mathbf{C})^{1/2}\,\mathbf{R}^T \end{aligned} \tag{3.2}$$

In addition to the tensor $\mathbf{C}$ the tensor

$$\mathbf{e} = \mathbf{F}^{-1T}\mathbf{S}^T\mathbf{C}\mathbf{S}\mathbf{F}^{-1} = {}^1/_2\mathbf{R}\,(\mathbf{I} - \mathbf{V}^{-2})\,\mathbf{R}^T = {}^1/_2\,(\mathbf{I} - \mathbf{F}^{-1T}\mathbf{S}^T\mathbf{S}\mathbf{F}^{-1}) \tag{3.3}$$

that agrees with the Almansi tensor for $\mathbf{S} = \mathbf{I}$ will be useful in later constructions.

A relationship connecting the rate of change of a tensor $\mathbf{e}$ with the total strain rate of the tensor $\mathbf{S}^\cdot$

$$\begin{aligned} &\mathbf{e}^\cdot + \mathbf{e}\,(\mathbf{F}^\cdot\mathbf{F}^{-1} - \mathbf{F}\mathbf{S}^{-1}\mathbf{S}^\cdot\mathbf{F}^{-1}) + (\mathbf{F}^{-1T}\mathbf{F}^{\cdot T} - \mathbf{F}^{-1T}\mathbf{S}^{\cdot T}\mathbf{S}^{-1T}\mathbf{F}^T)\,\mathbf{e} = \\ &{}^1/_2\,(\nabla\mathbf{v} + \nabla\mathbf{v}^T) - {}^1/_2\,(\mathbf{F}\mathbf{S}^{-1}\mathbf{S}^\cdot\mathbf{F}^{-1} + \mathbf{F}^{-1T}\mathbf{S}^{\cdot T}\mathbf{S}^{-1T}\mathbf{F}^T) \end{aligned} \tag{3.4}$$

is derived by direct differentiation of the tensor $\mathbf{e}$ with respect to time taking the kinematic relationship (1.5) into account.

The case is later considered when the elastic strain tensors are small compared with one, i.e., $\|\mathbf{C}\| \ll 1$, $\|\mathbf{e}\| \ll 1$, $\|\mathbf{V}\| = 1 + O(\|\mathbf{e}\|)$, and the rotations $\mathbf{R}$, the viscous $\mathbf{W}$ and the biological ω strains are arbitrary. This case is typical for the muscle material whose designation with respect to small stresses (as compared with the elastic moduli) causes finite displacements of the body parts.

In addition, we assume that (2.8) has the form

$$A = A^{\times}(\mathbf{C}, \mathbf{n}), \quad \mathbf{D}^{(P)} \equiv {}^1/_2(\mathbf{W}^{\cdot}\mathbf{W}^{-1} + \mathbf{W}^{-1}\mathbf{W}^{\cdot}) = \mathbf{\Phi}^{\times}(\mathbf{C}, \mathbf{n}); \qquad (3.5)$$
$$\det \mathbf{W} = 1, \quad \omega = \omega^{\times}(f(\mathbf{X}, t))$$

where $\mathbf{D}^{(P)}$ is the viscous strain rate tensor. These assumptions mean that the potential depends only on the elastic strains, and the viscous compressibility can be neglected: the biological strain is determined solely by the electrochemical signal f (central regulation), and the symmetric tensor $\mathbf{D}^{(P)}$, whose trace equals zero, is used as a measure of the viscous strain rate.

Selection of the tensor $\mathbf{D}^{(P)}$ as the measure of the viscous stain rate means the a priori assumption of the kind of function $\mathbf{\Psi}^{+}(\mathbf{V}, \mathbf{W}, \mathbf{n})$ in relationships (2.10), namely, the function $\mathbf{\Psi}^{+}$ is a solution of the equation

$$\mathbf{\Psi}^{+}\mathbf{W}^{-1} + \mathbf{W}^{-1}\mathbf{\Psi}^{+} = 2\mathbf{\Phi}^{\times}(\mathbf{C}, \mathbf{n})$$

By virtue of the positive definiteness of the symmetric tensor $\mathbf{W}$ the solution of this matrix equation exists and is unique for arbitrary right side.

Also assuming that the intermediate configuration $\varkappa_2$ is unloaded, i.e., the stress $\mathbf{T} = 0$ for $\mathbf{C} = 0$, while the isotropic potential $A^{\times}$ is a regular function in neighbourhood of $\mathbf{C} = 0$, we obtain to the accuracy of the terms $O(\mathbf{C}^3)$

$$\rho_0 A^{\times} = A_0 + {}^1/_2\lambda I_1 + \mu I_2 + \alpha_1 I_1 I_4 + {}^1/_2\alpha_2 I_4^2 + \alpha_3 I_5 \qquad (3.6)$$
$$I_1 = \operatorname{tr}\mathbf{C},\ I_2 = \operatorname{tr}\mathbf{C}^2,\ I_4 = \mathbf{nCn},\ I_5 = \mathbf{nC}^2\mathbf{n}$$

where $\lambda, \mu, \alpha_i = \text{const}$ and ρ_0 is the initial density of the material.

If the material is incompressible, then $I_1 \equiv 0$ and the potential is

$$\rho_0 A^{\times} = A_0 + \mu J_2 + {}^1/_2\alpha_2 J_4^2 + \alpha_3 J_5 \qquad (3.7)$$
$$J_2 = \operatorname{tr}(\mathbf{C}')^2,\ J_4 = n\mathbf{C}'n,\ J_5 = \mathbf{nC}'\mathbf{C}'\mathbf{n},\ \mathbf{C}' = \mathbf{C} - {}^1/_3\mathbf{I}\operatorname{tr}\mathbf{C}$$

Using the relation $\mathbf{e} = \mathbf{RCR}^T + O(\mathbf{e}^2)$ we obtain from (3.2) and (3.6) to the accuracy of first-order infinitesimal terms

$$\mathbf{T} = (\lambda I_1 + \alpha_1 I_4)\mathbf{I} + 2\mu\mathbf{e} + (\alpha_1 I_1 + \alpha_2 I_4)\,\mathbf{m}\otimes\mathbf{m} + \qquad (3.8)$$
$$\alpha_3(\mathbf{m}\otimes\mathbf{l} + \mathbf{l}\otimes\mathbf{m}) \quad (\mathbf{m} = \mathbf{Rn},\quad \mathbf{l} = \mathbf{em} = \mathbf{RCn})$$

In the case of an incompressible material

$$\mathbf{T} = -p_0\mathbf{I} + \rho_0\mathbf{R}(\partial A^{\times}/\partial\mathbf{C})\mathbf{R}^T = -p_0\mathbf{I} + 2\mu\mathbf{e} + \qquad (3.9)$$
$$\alpha_2 I_4(\mathbf{m}\otimes\mathbf{m} - {}^1/_3\mathbf{I}) + \alpha_3(\mathbf{m}\otimes\mathbf{l} + \mathbf{l}\otimes\mathbf{m} - {}^2/_3 I_4\mathbf{I})$$

As might have been expected, the Cauchy stress tensor in the model under consideration is defined by the tensor $\mathbf{e}$ dependent only on the elastic gradient $\mathbf{E}$ and the unit vector $\mathbf{m}$ giving the axis of transversal isotropy whose position in the actual configuration is determined by the orthogonal tensor $\mathbf{R}$.

We now consider the law of viscous flow (the second equation in (3.5)). We assume that $\mathbf{\Phi}^{\times} = \partial\delta/\partial\mathbf{C}$, i.e., $\mathbf{\Phi}^{\times}$ is the gradient of a sufficiently smooth scalar function $\delta = \delta(\mathbf{C}, \mathbf{n})$. In the neighbourhood of $\mathbf{e} = 0$ the latter has the form

$$\delta = {}^1/_2\beta_\lambda I_1^2 + \beta I_2 + \beta_1 I_1 I_4 + {}^1/_2\beta_2 I_4 + \beta_3 I_5 + O(\mathbf{e}^3)$$

The equalities

$$3\beta_\lambda + 2\beta + \beta_1 = 0,\ 3\beta_1 + \beta_2 + 2\beta_3 = 0$$

follow from the condition $\operatorname{tr}\mathbf{\Phi}^{\times} = 0$ of viscous incompressibility, and we obtain

$$\mathbf{\Phi}^{\times}(\mathbf{e}, \mathbf{m}) = \mathbf{R}\mathbf{\Phi}^{\times}(\mathbf{C}, \mathbf{n})\mathbf{R}^T = 2\beta(\mathbf{e} - {}^1/_3 I_1\mathbf{I}) + \qquad (3.10)$$
$$(\beta_2 I_4 - \nu I_1)(\mathbf{m}\otimes\mathbf{m} - {}^1/_3\mathbf{I}) + \beta_3(\mathbf{m}\otimes\mathbf{l} + \mathbf{l}\otimes\mathbf{m} - {}^2/_3 I_4\mathbf{I}) + O(e^2)$$

when we take them into account. Here and henceforth $\nu = {}^1/_3(\beta_2 + 2\beta_3)$. In the case of small elastic strains the kinematic relationship (3.4) obtains a substantial simplification. Indeed, as $\mathbf{e} \to \mathbf{0}$ we have

$$\mathbf{FS}^{-1}\mathbf{S}^{\cdot}\mathbf{F}^{-1} = \mathbf{G} + \mathbf{eG} - \mathbf{Ge} + O(\mathbf{e}^2)$$
$$\mathbf{F}^{\cdot}\mathbf{F}^{-1} = \mathbf{R}^{\cdot}\mathbf{R}^T + \mathbf{G} + O(\mathbf{e}),\quad \mathbf{G} = \mathbf{RS}^{\cdot}\mathbf{S}^{-1}\mathbf{R}^T$$

Then to the accuracy of terms $O(\mathbf{e}^2)$ the equality (3.4) is written in the form

$$\mathbf{e}^{\cdot} + \mathbf{e}\{\mathbf{R}^{\cdot}\mathbf{R}^T + {}^1/_2(\mathbf{G} - \mathbf{G}^T)\} - \{\mathbf{R}^{\cdot}\mathbf{R}^T + {}^1/_2(\mathbf{G} - \mathbf{G}^T)\}\mathbf{e} =$$
$${}^1/_2(\nabla\mathbf{v} + \nabla\mathbf{v}^T) - {}^1/_2(\mathbf{G} + \mathbf{G}^T)$$

taking into account that

$$\mathbf{R}^{\cdot}\mathbf{R}^T + {}^1/_2(\mathbf{G} - \mathbf{G}^T) = {}^1/_2(\nabla\mathbf{v} - \nabla\mathbf{v}^T) + O(\mathbf{e})$$
$${}^1/_2(\mathbf{G} + \mathbf{G}^T) = \mathbf{R}(\mathbf{D}^{(P)} + \mathbf{W}\mathbf{D}^{(B)}\mathbf{W}^{-1})\mathbf{R}^T$$

where $\mathbf{D}^{(P)}$ is defined by the second of relationships (3.5), and the symmetric deviator tensor

$$\mathbf{D}^{(B)} = {}^1/_2(\mathbf{U}_B^{\cdot}\mathbf{U}_B^{-1} + \mathbf{U}_B^{-1}\mathbf{U}_B^{\cdot}) = {}^1/_2\omega^{\cdot}\omega^{-1}(3\mathbf{n}\otimes\mathbf{n} - \mathbf{I})$$

is the tensor of the biological strain rate by definition, we obtain to $O(\mathbf{e}^2)$ accuracy

$$D\mathbf{e}/Dt = {}^1/_2(\nabla\mathbf{v} + \nabla\mathbf{v}^T) - \mathbf{\Phi}^{\times}(\mathbf{e}, \mathbf{m}) - \mathbf{R}\mathbf{W}\mathbf{D}^{(B)}\mathbf{W}^{-1}\mathbf{R}^T \tag{3.11}$$

($D\mathbf{e}/Dt$ is the Jaumann objective derivative).

4. **Example.** We consider the quasistatic axisymmetric problem /9, 10, 19, 20/ of the strain of a circular thick-walled cylinder from an ideal biologically active viscoelastic material subjected to internal pressure. We assume that the axis of transversal isotropy of the material is directed along the tangent perpendicular to the cylinder axis. This case is characteristic for blood vessels whose walls contain smooth muscle with fibres in the circumferential direction. We neglect mass forces, we consider the material as incompressible, the elastic strain small, and the viscous and biological finite. We take the reference configuration of the cylinder natural, and the stress and displacements therein equal to zero.

We select a cylindrical R, φ, Z coordinate system with the Z-axis directed along the cylinder axis. It is assumed that the axial displacements equal zero, i.e., all the sections $Z = \text{const}$ are under plane strain conditions. Because of the axial symmetry there are also no angular displacements.

Let r, φ, z be the coordinates of a material particle in the reference configuration, a the radius of the inner, and b the radius of the outer surface of the cylinder for $t = 0$. Then the domain of definition of the solution has the form $\{a \leqslant r \leqslant b, t \geqslant 0\}$. Let $\sigma_R, \sigma_\varphi, \sigma_Z$ be the physical components of the Cauchy tensor in an Euler R, φ, Z coordinate system, $v = \partial R(r, t)/\partial t$ is the particle radial velocity, e_R, e_φ, e_Z are non-zero diagonal elements of the matrix of physical components of the elastic strain tensor $\mathbf{e}$. We consider the quantity $\{v, e_R, e_\varphi, e_Z, R, p_0\}$ as a vector of the solution.

The initial conditions of the problem are

$$v = e_R = e_\varphi = e_Z = p_0 = 0 \tag{4.1}$$
$$R(r, 0) = r, \; a \leqslant r \leqslant b, \; t = 0$$

The boundary conditions corresponding to the pressure $-p_1(t)$, $p_1 > 0$ on the inner and $-p_2(t)$, $p_2 > 0$ on the outer cylinder surfaces, will be the conditions

$$\sigma_R(e_R, e_Z, p_0) = -p_1(t), \; r = a, \; t > 0 \tag{4.2}$$
$$\sigma_R(e_R, e_Z, p_0) = -p_2(t), \; r = b, \; t > 0$$

Taking account of the symmetry of the problem we have

$$D_R^{(B)} = D_Z^{(B)} = -{}^1/_2 D_\varphi^{(B)} = -\omega^{\cdot}\omega^{-1}, \; \mathbf{R} = \mathbf{I}, \; F_R = \partial R/\partial r$$
$$F_\varphi = R/r, \; F_Z = 1, \; F_{ij} = 0, \; i \neq j$$

where ω is a known function of r, t.

We use the notation

$$\alpha = {}^1/_3(\alpha_2 + 2\alpha_3), \; \nu = {}^1/_3(\beta_2 + 2\beta_3), \; \gamma(r, t) = \omega^{\cdot}(r, t)\,\omega^{-1}(r, t)$$

The complete system of equations is written in the form

$$\partial\sigma_R/\partial R + (\sigma_R - \sigma_\varphi)/R = 0 \tag{4.3}$$
$$e_R^{\cdot} = \partial v/\partial R - 2\beta e_R + \nu e_\varphi + \gamma, \quad e_\varphi^{\cdot} = v/R - 2\beta_0 e_\varphi - 2\gamma \tag{4.4}$$
$$e_Z^{\cdot} = -2\beta e_Z + \nu e_\varphi + \gamma \;\; (\beta_0 = \beta + \nu)$$
$$\partial v/\partial R + v/R = 0, \; R^{\cdot} = v \tag{4.5}$$
$$\sigma_R = -p_0 + 2\mu e_R - \alpha e_\varphi, \; \sigma_\varphi = -p_0 + 2(\mu + \alpha) e_\varphi, \tag{4.6}$$
$$\sigma_Z = -p_0 + 2\mu e_Z - \alpha e_\varphi$$

Here (4.3) is the equilibrium Eq.(4.4), is (3.12) for compatibility of the strains and velocities converted taking (3.10) into account, (4.5) is the condition of incompressibility and the definition of the material particle velocity, and (4.6) is the relation between the physical components of the Cauchy stress tensor and the small elastic strains.

From the incompressibility condition (4.5) and the initial data (4.1) it follows that ($c(t)$ is a still unknown function of time)

$$v = {}^1/_2 c'(t)/R(r, t), \quad c'(0) = 0 \tag{4.7}$$

Setting $c(0) = 0$ without loss of generality, we obtain from the second equation in (4.6) and the initial condition

$$R(r, t) = \sqrt{r^2 + c(t)} \tag{4.8}$$

Taking account of the initial data (4.1) and (4.7) and (4.8), Eqs.(4.4) yield

$$\begin{aligned} &e_\varphi(r, t) = \varphi(r, t) - 2\beta_0 K_0 \varphi - 2K_0\gamma \\ &e_R(r, t) = -\varphi(r, t) + (\beta K + \beta_0 K_0)\varphi + K_0\gamma \\ &\varphi(r, t) = \ln(r^{-1}R(r, t)) = {}^1/_2 \ln(1 + r^{-2}c(t)) \\ &K_0\varphi \equiv \int_0^t \exp[2\beta_0(\xi - t)]\varphi(r, \xi)\,d\xi, \quad K\varphi \equiv \int_0^t \exp[2\beta(\xi - t)]\varphi(r, \xi)\,d\xi \end{aligned} \tag{4.9}$$

Taking account of (4.6), (4.8) and (4.9) the equilibrium equation reduces to

$$\begin{aligned} &\partial\sigma_R/\partial r = 2\mu r^{-1}\exp[-2\varphi(r, t)]\,\{(1 + \eta)\varphi(r, t) - [\beta K + (1 + \\ &\quad 2\eta)\beta_0 K_0]\varphi - (1 + 2\eta)K_0\gamma\}, \quad \eta = 1 + 3\alpha/(2\mu) \end{aligned}$$

Integrating with respect to the radius and taking the boundary conditions (4.2) on the inner and outer surfaces into account, we arrive at a non-linear integral equation

$$\begin{aligned} &(1 + \eta)\psi(t) - (\beta K + (1 + 2\eta)\beta_0 K_0)\psi = {}^1/_2\mu^{-1}\Delta p + (1 + 2\eta)K_0 G \\ &\psi(t) = \int_a^b \exp[-2\varphi(r, t)]\,\varphi(r, t) r^{-1}\,dr \\ &G(t) = \int_a^b \exp[-2\varphi(r, t)]\,\gamma(r, t)\,r^{-1}\,dr \end{aligned} \tag{4.10}$$

where $\Delta p = p_1(t) - p_2(t)$ is the pressure drop on the vessel wall.

If the displacements are small together with the biological strain, i.e. $c(t) \ll a^2$, $\gamma(r, t) \ll 1$, then to the accuracy of terms $o(c/a^2)$

$$\varphi = \frac{c(t)}{2r^2}, \quad \psi = \frac{b^2 - a^2}{4a^2b^2}c(t), \quad G = \int_a^b r^{-1}\gamma(r, t)\,dr$$

and after differentiation with respect to the time we obtain from (4.10)

$$\begin{aligned} &c'' + 2mc' = q(t) \\ &m = \beta + {}^1/_2\nu/(1 + \eta), \quad q(t) = [g''(t) + 2(\beta + \beta_0)g'(t) + \\ &\quad 4\beta\beta_0 g(t)]/(1 + \eta) \end{aligned}$$

The solution of this equation that satisfies the initial conditions $c(0) = c'(0) = 0$ is the following:

$$c(t) = \int_0^t \exp[2m(\xi - t)]\int_0^\xi q(s)\,ds\,d\xi \tag{4.11}$$

For times $t \ll 1/\beta_0$ when viscosity effects can be neglected, we have the bioelastic approximation /1/.

It is seen from solution (4.11) that the condition $q(t) \leqslant 0$ is a condition for the appearance of the Bayliss effect /21/: the diminution in the clearance of a vessel due to feedback for a quasistatic growth of the intravascular pressure. If the quantity $\gamma(r, t) = \gamma(t) < 0$, i.e., is identical at all points of the vessel wall, then the condition $q(t) \leqslant 0$ reduces to the equality

$$|\gamma(t)| \geqslant [\mu(1 + 2\eta)\ln(b/a)]^{-1}(\Delta p'(t) + 2\beta\Delta p(t))$$

connecting the rate of change of the biological strain with the current value of the pressure drop and its rate of change.

REFERENCES

1. NIKITIN L.V., A model of a bioelastic body, Izv. Akad. Nauk SSSR, Mekhan., Tverd. Tela, 3, 1971.
2. USIK P.I., Continuum mechanochemical model of muscel tissue, PMM, 37, 3, 1973.
3. REGIRER S.A., Lectures on Biomechanics, Pt.1, Izd. Moskov. Gosudarst. Univ., 1980.
4. NIKITIN N.L., Model of muscle tissue with a variable number of working fibers, Mekhan. Komposit. Materialov, 1, 1980.

5. REGIRER S.A. and TSATURYAN A.K., Fundamental problems of the mechanics of muscle contraction, Modern Biomechanics Problems /in Russian/, 1, Zinatne, Riga, 1983.
6. KLOCHKOV B.N., On taking account of finite strains in the model of muscle, Proceedings the 3rd All-Union Conf. on Biomechanics Problems, 1, Izd. Inst. Mekhan. Polimerov, Akad. Nauk Latv. SSR, Riga, 1983.
7. PETROV N. and NIKITIN L., Application of thermodynamics of open systems to the description of muscle operation, Biomekhanika (Sofiya), 4, 1977.
8. AKHUNDOV M.B., RABOTNOV YU.N. and SUVOROVA YU.V., Model of a deformable body with reactions and its application to dynamical biomechanics problems, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 6, 1985.
9. NIKITIN L.V. and KHAYUTIN V.M., Theory of measurement of the hydraulic drag of vessels subjected to control signals, Fiziolog. Zhurn. 48, 8, 1962.
10. REGIRER S.A., RUTKEVICH I.M. and USIK P.I., A model of vascular tonus, Mekhan. Polimerov, 4, 1975.
11. HILL A., Mechanics of Muscle Contraction, Mir, Moscow, 1972.
12. BENDALL J., Muscles, Molecules, and Motion, Mir, Moscow, 1970.
13. GURFINKEL' V.S. and LEVIK YU.S., Skeletal Muscle: Structure and Function. Nauka, Moscow, 1985.
14. IL'YUSHIN A.A. and POBEDRYA B.E., Principles of the Mathematical Theory of Thermoviscoelasticity. Nauka, Moscow, 1970.
15. TRUESDELL C., Initial Course in the Rational Mechanics of Continuous Media, Mir, Moscow, 1975.
16. ARUTYUNYAN N.KH. and NAUMOV V.E., Boundary value problem of the theory of viscoelastic plasticity for a growing body subjected to ageing, PMM, 48, 1, 1984.
17. LUR'E A.I., Non-linear Elasticity Theory. Nauka, Moscow, 1980.
18. KONDAUROV V.I., Equations of relaxation type for viscoelastic media with finite strain, PMM, 49, 5, 1985.
19. REGIRER S.A. and RUTKEVICH I.M., Wave motions of a fluid in pipes of viscoelastic material: Small amplitude waves, Izv. Akad. Nauk SSSR, Mekhan., Zhidk. Gaza, 1, 1975.
20. PURINA B.A. and KAS'YANOV B.A., Biomechanics of Large-Scale Blood Vessels, Zinatne, Riga, 1980.
21. KONRADI G.P., Regulation of Vascular Tonus, Nauka, Leningrad, 1973.

Translated by M.D.F.

PMM U.S.S.R.,Vol.51,No.3,pp.353-356,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

EQUILIBRIUM AND STABILITY OF NON-LINEARLY ELASTIC BODIES WITH CAVITIES CONTAINING FLUID*

V.A. YEREMEYEV and L.M. ZUBOV

Boundary conditions are formulated on the surface of a cavity filled with a compressible fluid or gas for the equilibrium problem of an elastic body experiencing large deformations. A formulation is given of the stability problem for the equilibrium of a non-linearly elastic body with fluid inclusions. The stability problem is solved for a thick-walled closed spherical shell filled with gas and loaded by external pressure.

1. We consider an elastic body occupying a volume v in the reference configuration. Let the boundary of the domain v consist of $m+1$ closed surfaces $\sigma, \sigma_1, \ldots, \sigma_m$, where σ_k $(k = 1, 2, \ldots m)$ are surfaces of simply-connected cavities, and $\sigma = \sigma' \bigcup \sigma''$ is the outer boundary enclosing the body with the cavities. Each cavity is filled entirely with a compressible barotropic homogeneous liquid or gas. The body is deformed under the action of external forces distributed over parts of the surface σ'. Displacements are given on the surface σ''. We neglect the action of the mass forces. The pressure of the liquid is constant in each of

**Prikl.Matem.Mekhan.*,51,3,453-457,1987